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Gale's feasibility theorem and max–flow problems in a continuous network

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1 Introduction

Gale's feasibility theorem was originally formulated on a discrete network in [4] . It is known as the "Supply - Demand Theorem" in a special case and gives a necessary and sufficient condition for an existence of feasible flows.

In [11], we established a continuous version of the theorem on a Euclidean domain. There are several formulations of continuous networks. Our problem is formulated in a framework of a continuous network introduced by [6] and [13].

In contrast with discrete cases, our continuous version is essentially related with the boundedness of constraints of flows. However, we can deal with a certain special case with unbounded constraints such as problems in [5]. In the present paper, we investigate the continuous version of Gale's feasibility theorem in a more general setting which can be applied to problems with a certain class of unbounded constraints of flows.

Let us recall our formulation of continuous networks and state a continuous version of the Supply - Demand Theorem. As for a discrete version, one can refer to Ford and Fulkerson [3]. In this discussion, we assume that all functions and sets are sufficiently smooth. Let Ω be a bounded domain of n-dimensional Euclidean space R^n and $\partial\Omega$ be the boundary. Let A, B be disjoint subsets of $\partial\Omega$ which are regarded as a source and a sink. In our continuous network, every flow is represented by a vector field and every feasible flow σ satisfies the capacity constraint:

$$\sigma(x) \in \Gamma(x) \text{ for all } x \in \Omega,$$

where Γ is a set-valued mapping from Ω to R^n. We call Ω with this capacity constraint a continuous network.

Furthermore, every cut is identified with a subset of Ω in our network. Let S be a cut and ν^S be the unit outer normal to S. Then the cut capacity $C(S)$ is defined by

$$C(S) = \int_{\Omega\cap\partial S} \beta(-\nu^S(x), x)ds(x),$$

where

$$\beta(v,x) = \sup_{w\in\Gamma(x)} v\cdot w$$

for $v \in R^n$ and ds is the surface element. If the capacity constraint is isotropic, that is, $\Gamma(x) = \{w \in R^n;\ |w| \leq c(x)\}$ with some nonnegative function $c(x)$, then

$$C(S) = \int_{\Omega\cap\partial S} c(x)ds(x).$$

Let a, b be real-valued functions on A, B respectively and let ν be the unit outer normal to Ω. Then the problem of supply-demand is stated as follows:

$$(SD) \qquad \text{Find } \sigma \text{ such that } \sigma(x) \in \Gamma(x) \text{ for all } x \in \Omega, \operatorname{div}\sigma = 0 \text{ on } \Omega,$$
$$\sigma\cdot\nu = 0 \text{ on } \partial\Omega - (A\cap B), -\sigma\cdot\nu \leq a \text{ on } A,\ \sigma\cdot\nu \geq b \text{ on } B.$$

The Supply-Demand theorem assures that (SD) has a solution if and only if

$$(G) \qquad C(S) \geq \int_{B\cap\partial S} bds - \int_{A\cap\partial S} ads \ \text{ for each cut } S.$$

This can be proved by the aid of a continuous version of max-flow min-cut theorem under certain additional conditions, if $\cup_{x\in\Omega}\Gamma(x)$ is bounded. Moreover, it is also proved by a method used in [9] and [12], which is based on a generalized Hahn-Banach Theorem.

In the next section, we give a concrete formulation of our problem in a general form including (SD) as its special case, and investigate a necessary and sufficient condition under which the problem has a solution. In §3, we are concerned with an equivalence between the feasibility theorem and a max-flow min-cut theorem.

2 Problem setting and a main theorem

Let Ω be a bounded domain in n-dimensional Euclidean space R^n with Lipschitz boundary $\partial\Omega$. Let H_{n-1} be the $n-1$-dimensional Hausdorff measure. Then H_{n-1} on $\partial\Omega$ can be identified with the surface measure on $\partial\Omega$. We note that the unit outer normal ν to Ω is defined and essentially bounded measurable on $\partial\Omega$ with respect to H_{n-1}. Let Γ be a set-valued mapping from Ω to R^n which satisfies the following two conditions:

(H1) $\Gamma(x)$ is a compact convex set containing 0 for all $x \in \Omega$.

(H2) Let $\varepsilon > 0$ and Ω_0 be a compact subset of Ω. Then there is $\delta > 0$ such that $\Gamma(x) \subset \Gamma(y) + B(0,\varepsilon)$ if $x, y \in \Omega_0$ and $|x-y| < \delta$.

In what follows, we assume that each feasible flow is represented by an essentially bounded vector field σ on Ω satisfying the following capacity constraints:

$$\sigma(x) \in \Gamma(x) \quad \text{for a.e. } x \in \Omega.$$

Furthermore if $\operatorname{div} \sigma \in L^n(\Omega)$, then $\sigma \cdot \nu$ can be defined as a function in $L^\infty(\partial\Omega)$ in a weak sense by [7].

Let X be a nonempty subset of $L^n(\Omega) \times L^\infty(\partial\Omega)$. Then for the triple (Ω, Γ, X), our problem is stated as follows:

(P) Find $\sigma \in L^\infty(\Omega; R^n)$ such that $\sigma(x) \in \Gamma(x)$ for a.e. $x \in \Omega, (-\operatorname{div} \sigma, \sigma \cdot \nu) \in X$.

Problem (SD) considered in §1 can be written in this form with $X = \{(F, f);\ F = 0, f \geq -a \text{ on } A, f \geq b \text{ on } B\}$.

To specify the class of cuts, we consider the space $BV(\Omega)$ of functions of bounded variation on Ω, and a Sobolev space $W^{1,1}(\Omega)$ which is regarded as a subspace of $BV(\Omega)$:

$$\begin{aligned} BV(\Omega) &= \{u \in L^1(\Omega);\ \nabla u \text{ is a Radon measure} \\ &\qquad \text{of bounded variation on } \Omega\}, \\ W^{1,1}(\Omega) &= \{u \in L^1(\Omega)\ ;\ \nabla u \in L^1(\Omega; R^n)\}, \end{aligned}$$

where $\nabla u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ is understood in the sense of distribution. It is known that $BV(\Omega) \subset L^{n/(n-1)}(\Omega)$ and the trace γu is determined as a function in $L^1(\partial\Omega)$ for each $u \in BV(\Omega)$.

We denote the characteristic function of a subset S of Ω by χ_S and set

$$Q = \{S \subset \Omega;\ \chi_S \in BV(\Omega)\}.$$

Let $S \in Q$. Then the reduced boundary $\partial^* S$ of S is the set of all $x \in \partial S$ where Federer's normal $\nu^S = \nu^S(x)$ to S exists. (One can refer to [8] for the details.) It is known that $\partial^* S$ is a measurable set with respect to both the measure of total variation of $|\nabla \chi_S|$ and H_{n-1}, $|\nabla \chi_S|(R^n - \partial^* S) = 0$ and $|\nabla \chi_S|(E) = H_{n-1}(E)$ for each $|\nabla \chi_S|$-measurable subset E of $\partial^* S$. Then [8, Theorem 6.6.2] implies that $\gamma \chi_S = \chi_{\partial^* S \cap \partial\Omega}$ H_{n-1}-a.e. on $\partial\Omega$.

Let $\beta(\cdot, x)$ be the support functional of $\Gamma(x)$ as defined in §1. If **(H1)** and **(H2)** holds, then β is continuous and nonnegative. Accordingly, in the case, replacing ds by H_{n-1} and ∂S by $\partial^* S$, we can define the cut capacity as follows:

$$C(S) = \int_{\Omega \cap \partial^* S} \beta(-\nu^S(x), x) dH_{n-1}.$$

Let $\nabla u/|\nabla u|$ be the Radon-Nikodym derivative of ∇u with respect to $|\nabla u|$ and set

$$\psi(u) = \int_\Omega \beta(\nabla u/|\nabla u|, x)d|\nabla u|(x)$$

for $u \in BV(\Omega)$. Then $C(S) = \psi(\chi_S)$.

If we assume the following **(H2′)** instead of **(H2)**, then we can define $\psi(u)$ only for $u \in W^{1,1}(\Omega)$:

$$\textbf{(H2}'\textbf{)} \qquad \{(x,w);\ w \in \Gamma(x), x \in \Omega\} \text{ is measurable,}$$

Now we set $L_{(F,f)}(u) = \displaystyle\int_\Omega Fudx + \int_{\partial\Omega} f\gamma udH_{n-1}$ and consider the following condition under **(H1)** and **(H2)**:

$$\textbf{(C)} \qquad \psi(u) \geq \inf_{(F,f)\in X} L_{(F,f)}(u) \text{ for all } u \in BV(\Omega).$$

We note that u can be replaced by characteristic functions of sets in Q in some cases. When **(H2′)** is assumed instead of **(H2)**, replacing $BV(\Omega)$ in **(C)** by $W^{1,1}(\Omega)$ we consider

$$\textbf{(C}'\textbf{)} \qquad \psi(u) \geq \inf_{(F,f)\in X} L_{(F,f)}(u) \text{ for all } u \in W^{1,1}(\Omega).$$

Now we have

PROPOSITION 2.1. *If* **(H1)**, **(H2)** *hold and* **(P)** *has a solution, then* **(C)** *is satisfied. Similarly, If* **(H1)**, **(H2′)** *hold and* **(P)** *has a solution, then* **(C′)** *is satisfied.*

Proof. Assume **(H1)** and **(H2)**. Let σ be a solution of **(P)** and $u \in BV(\Omega)$. Then by Green's formula stated below and [10, Lemma 2.6],

$$\begin{aligned}\psi(u) \geq (\sigma\nabla u)(\Omega) &= \int_{\partial\Omega} \sigma\cdot\nu\gamma udH_{n-1} - \int_\Omega u\mathrm{div}\,\sigma dx \\ &\geq \inf_{(F,f)\in X} L_{(F,f)}(u).\end{aligned}$$

When **(H2′)** is assumed instead of **(H2)**, the inequality is similarly proved for $u \in W^{1,1}(\Omega)$. □

The following Green's formula is due to [7, Proposition 1.1]:

LEMMA 2.2. *Let* $\sigma \in L^\infty(\Omega; R^n)$ *such that* div $\sigma \in L^n(\Omega)$ *and* $u \in BV(\Omega)$. *Then the distribution* $(\sigma\nabla u)$ *defined by*

$$(\sigma\nabla u)(\varphi) = -\int_\Omega u\nabla\varphi\cdot\sigma dx - \int_\Omega u\varphi\mathrm{div}\,\sigma dx$$

for $\varphi \in C_0^\infty(\Omega)$ is a bounded measure. Furthermore

$$(\sigma\nabla u)(\Omega) + \int_\Omega u\mathrm{div}\ \sigma dx = \int_{\partial\Omega} \gamma u\sigma \cdot \nu dH_{n-1}$$

holds.

We note that $(\sigma\nabla u)(\Omega) = \int_\Omega \sigma \cdot \nabla u dx$ for $u \in W^{1,1}(\Omega)$.

The following lemma is regarded as a continuous version of max-flow min-cut theorem, which is due to [13].(The proof is in [10].)

LEMMA 2.3. *Assume that $\cup_{x\in\Omega}\Gamma(x)$ is bounded and* **(H1)**, **(H2′)** *hold. Then*

$$\begin{aligned} &\sup\{\lambda;\ \text{there is a feasible flow } \sigma \\ &\qquad\qquad \text{such that } (-\mathrm{div}\,\sigma, \sigma\cdot\nu) = \lambda(F,f)\} \\ = \ &\inf\{\psi(u)/L_{(F,f)}(u);\ u \in W^{1,1}(\Omega) \\ &\qquad\qquad \text{such that } L_{(F,f)}(u) > 0\}. \end{aligned}$$

Furthermore if **(H2)** *holds, then this equals*

$$\inf\{C(S)/L_{(F,f)}(\chi_S);\ S \in Q \text{ such that } L_{(F,f)}(\chi_S) > 0\}$$

This lemma implies

LEMMA 2.4. (1) *For each $F \in L^n(\Omega)$, there is $\sigma \in L^\infty(\Omega; R^n)$ such that $-\mathrm{div}\,\sigma = F$ a.e. on Ω.*

(2) *Assume that there is a constant k, independent of u, satisfying $\inf_{c\in R} \int_{\partial\Omega} |\gamma u - c| dH_{n-1} \leq k\|\nabla u\|_\Omega$ for all $u \in BV(\Omega)$. Then for each $F \in L^n(\Omega)$ and $f \in L^\infty(\partial\Omega)$, there is $\sigma \in L^\infty(\Omega; R^n)$ such that $-\mathrm{div}\,\sigma = F$ a.e. on Ω and $\sigma\cdot\nu = f$ H_{n-1}-a.e. on Ω if and only if (F,f) satisfies the conservation law:*

$$\int_\Omega F dx + \int_{\partial\Omega} f dH_{n-1} = 0$$

Proof. (1) First assume that $\int_\Omega F dx = 0$. To prove the existence of σ_0 such that $-\mathrm{div}\,\sigma_0 = F$ a.e. on Ω, it is sufficient to show that the supremum

$$\begin{aligned} &\sup\{t \geq 0\ ;\ -\mathrm{div}\,\sigma = tF \text{ a.e. on } \Omega,\ \sigma\cdot\nu = 0\ H_{n-1}\text{-a.e. on } \partial\Omega \\ &\qquad \text{for some } \sigma \in L^\infty(\Omega; R^n) \text{ with } \|\sigma\|_\infty \leq 1\} \end{aligned}$$

is positive. Since it is equal to

$$\inf\{H_{n-1}(\Omega\cap\partial^* S)/\int_S Fdx\ ;\ \int_S Fdx>0,\ S\subset\Omega,\chi_S\in BV(\Omega)\}$$

by the preceding lemma, we shall prove that the infimum is positive. According to [8, p.303] there is a positive constant k_0 such that $\min(m_n(S),m_n(\Omega-S))\le k_0H_{n-1}(\Omega\cap\partial^*S)^{n/(n-1)}$, where m_n denotes the Lebesgue measure on R^n. Since

$$\int_S Fdx\le(\int_S 1dx)^{(n-1)/n}\cdot(\int_S |F|^n dx)^{1/n}\le\|F\|_n(m_n(S))^{(n-1)/n}$$

and

$$\begin{aligned}\int_S Fdx=\int_{\Omega-S}-Fdx&\le(\int_{\Omega-S}1dx)^{(n-1)/n}\cdot(\int_{\Omega-S}|F|^n dx)^{1/n}\\&\le\|F\|_n(m_n(\Omega-S))^{(n-1)/n},\end{aligned}$$

we can conclude that

$$\int_S Fdx\le k_1H_{n-1}(\Omega\cap\partial^*S)$$

with $k_1=\|F\|_n k_0^{(n-1)/n}$ for all $S\in Q$. It follows that the infimum is not less than $1/k_1$.

Finally in case of $\int_\Omega Fdx\ne0$, consider σ_1 such that $\operatorname{div}\sigma_1$ equals constantly $-\int_\Omega Fdx/m_n(\Omega)$, σ_2 such that $\operatorname{div}\sigma_2=-F+\int_\Omega Fdx/m_n(\Omega)$ and set $\sigma_0=\sigma_1+\sigma_2$. Then $\operatorname{div}\sigma_0=F$. This completes the proof of (1).

(2) There is $\sigma_1\in L^\infty(\Omega;R^n)$ such that $-\operatorname{div}\sigma_1=F$ a.e. on Ω by (1). Setting $f_0=-\sigma_1\cdot\nu+f$ and show that there is $\sigma_2\in L^\infty(\Omega;R^n)$ such that $\operatorname{div}\sigma_2=0$ a.e. on Ω and $\sigma_2\cdot\nu=f_0$ H_{n-1}-a.e. on $\partial\Omega$. Since $\int_{\partial\Omega}f_0dH_{n-1}=0$ by Green's formula,

$$\begin{aligned}\|\nabla u\|_\Omega&\ge k^{-1}\inf_{c\in R}\int_{\partial\Omega}|\gamma u-c|dH_{n-1}\\&\ge k^{-1}\|f_0\|^{-1}_{L^\infty(\partial\Omega)}\inf_{c\in R}\int_{\partial\Omega}f_0(\gamma u-c)dH_{n-1}\\&=k^{-1}\|f_0\|^{-1}_{L^\infty(\partial\Omega)}\int_{\partial\Omega}f_0\gamma u dH_{n-1}.\end{aligned}$$

It follows again from the preceding lemma that

$$\begin{aligned}&\sup\{\lambda;\ \sigma\in L^\infty(\Omega;R^n),\|\sigma\|_\infty\le1,(-\operatorname{div}\sigma,\sigma\cdot\nu)=\lambda(0,f_0)\}\\=\ &\inf\{\|\nabla u\|_\Omega/\int_{\partial\Omega}f_0\gamma u dH_{n-1};\ u\in W^{1,1}(\Omega)\text{ such that }\int_{\partial\Omega}f_0\gamma u dH_{n-1}>0\}\end{aligned}$$

is positive. This implies that there is $\sigma_2\in L^\infty(\Omega;R^n)$ such that $\operatorname{div}\sigma_2=0$ a.e. on Ω and $\sigma_2\cdot\nu=f_0$ H_{n-1}-a.e. on $\partial\Omega$. Hence $\sigma=\sigma_1+\sigma_2$ satisfied the desired condition. This completes the proof. $\square$

Let ω be an open subset Ω with Lipschitz boundary. Then we call ω an admissible set if for each $F \in L^n(\omega)$ and $f \in L^\infty(\partial\omega)$ satisfying the conservation law, there is $\sigma \in L^\infty(\omega; R^n)$ such that $-\operatorname{div}\sigma = F$ a.e. on ω and $\sigma \cdot \nu = f$ H_{n-1}-a.e. on $\partial\omega$. If there is a constant k such that

$$\min(H_{n-1}(\partial\omega \cap \partial^* S), H_{n-1}(\partial\omega - \partial^* S)) \leq k H_{n-1}(\omega \cap \partial^* S)$$

for all $S \subset \omega$ with $\chi_S \in BV(\omega)$, then ω is admissible, since the inequality is equivalent with that in Lemma 2.4 (2) by [8, Theorem 6.5.2].

Now we state the converse of Proposition 2.1.

THEOREM 2.5. *Assume that* **(H1)** *and* **(H2′)** *holds. Then condition* **(C′)** *implies that* **(P)** *has a solution if one of the following two conditions is satisfied:*

(H3) $\cup_{x\in\Omega}\Gamma(x)$ *is bounded,* X *is weakly* closed convex and the projection of* X *to* $L^n(\Omega)$ *is bounded.*

(H4) X *is weakly* compact convex and there is an open subset* ω *of* Ω *such that* $\cup_{x\in\omega}\Gamma(x)$ *is bounded,* $\Gamma(x) = R^n$ *for all* $x \in \Omega - \omega$, Ω *has the Lipschitz boundary and* $\Omega - \overline{\omega}$ *is admissible.*

Proof. (1) First assume **(H3)** in addition to **(H1)**, **(H2′)** and **(C′)**. Let $U = L^1(\Omega; R^n) \times L^1(\partial\Omega)$ and $U^* = L^\infty(\Omega; R^n) \times L^\infty(\partial\Omega)$. Then (U, U^*) is regarded as a paired space with the bilinear form defined by $\langle(v,\phi),(w,f)\rangle = \int_\Omega v \cdot w dx + \int_{\partial\Omega} \phi f dH_{n-1}$ for $(v,\phi) \in U$ and $(w,f) \in U^*$.

Furthermore let $V = W^{1,1}(\Omega)$ and $V^* = L^n(\Omega) \times L^\infty(\partial\Omega)$. Since $W^{1,1}(\Omega) \subset L^{n/(n-1)}(\Omega)$ and the trace γu of $u \in W^{1,1}(\Omega)$ is in $L^1(\partial\Omega)$,

$$\langle\langle u, (F,f)\rangle\rangle = \int_\Omega F u dx + \int_{\partial\Omega} f \gamma u dH_{n-1}$$

defines a bilinear form on $V \times V^*$. Since $\{\gamma u;\ u \in W^{1,1}(\Omega)\} = L^1(\partial\Omega)$, (V, V^*) is also a paired space with the bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$. We consider the weak topologies on U, U^*, V, V^* by their pairings.

Let $\rho(v,\phi) = \int_\Omega \beta(v(x), x) dx$ for $(v,\phi) \in U$. We note that ρ is convex and positively homogeneous on U and constant with respect to the second argument.

On the other hand, if $u_1, u_2 \in V$ and $(\nabla u_1, \gamma u_1) = (\nabla u_2, \gamma u_2)$, then $u_1 = u_2$ a.e. on Ω, so that $L_{(F,f)}(u)$ is regarded as a function of $(\nabla u, \gamma u)$. Hence we can set

$$\Phi(\nabla u, \gamma u) = \inf_{(F,f)\in X} L_{(F,f)}(u).$$

Then Φ is a concave and positively homogeneous functional defined on the subspace $W = \{(\nabla u, \gamma u);\ u \in V\}$ of U. It follows from **(C′)** that there is a linear functional ξ on U such that $\xi \leq \rho$ on U and $\xi \geq \Phi$ on W.

The continuity of ξ follows from the boundedness of $\cup_{x\in\Omega}\Gamma(x)$. In fact, letting $M = \sup\{|w|;\ w \in \cup_{x\in\Omega}\Gamma(x)\}$, we have

$$\xi(v,\phi) \leq \rho(v,\phi) = \int_\Omega \beta(v(x),x)dx = M\|v\|_{L^1(\Omega;R^n)}.$$

Hence there is $(\sigma_0, \mu_0) \in U^*$ such that $\xi(v,\phi) = \displaystyle\int_\Omega \sigma_0 \cdot v dx + \int_{\partial\Omega} \phi\mu_0 dH_{n-1}$. However, since $\rho(v,\phi)$ is independent of ϕ, we conclude that $\mu_0 = 0$.

Now to show that σ_0 is a solution of **(P)**, we set

$$K = \{\sigma \in L^\infty(\Omega; R^n);\ \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega\}$$

and assume that $\sigma_0 \notin K$. Then there is a measurable set Ω_1 such that $\sigma_0(x) \notin \Gamma(x)$ for all $x \in \Omega_1$ and the Lebesgue measure $m_n(\Omega_0)$ of Ω_0 is positive. By applying a measurable selection theorem (cf. [2]) to $\tilde{\Gamma}(x) = \{w \in R^n;\ \sigma_0 \cdot w > \beta(w,x),\ |w| = 1\}$, there is $\eta \in L^\infty(\Omega_1, R^n)$ such that $\displaystyle\int_{\Omega_1} \sigma_0 \cdot \eta dx > \int_{\Omega_1} \beta(\eta, x)dx$. This is a contradiction since

$$\xi(\tilde{\eta}, 0) = \int_\Omega \sigma_0 \cdot \tilde{\eta} < \int_\Omega \beta(\tilde{\eta}, x)dx = \rho(\tilde{\eta}, 0)$$

for $\tilde{\eta} = \eta$ on Ω_1 and $\tilde{\eta} = 0$ on $\Omega - \Omega_1$.

Next, let P_X be the projection of X to $L^n(\Omega)$ and let $L = \sup_{F\in P_X} \|F\|_{L^n(\Omega)}$. By **(H3)**, L is finite. Since

$$\xi(\nabla u, \gamma u) = \int_\Omega \sigma_0 \cdot \nabla u dx \geq \Phi(\nabla u, \gamma u) = \inf_{F\in P_X} \int_\Omega F u dx$$

for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega \sigma_0 \cdot \nabla u dx \geq -L \cdot \|u\|_{L^{n/(n-1)}(\Omega)}.$$

This means that $\operatorname{div} \sigma_0 \in L^n(\Omega)$. Hence $(\operatorname{div} \sigma_0, \sigma_0 \cdot \nu) \in V^*$.

We can show that X is a closed convex set of V^* with respect to the weak topology of our pairing by **(H3)** so that if $(-\operatorname{div} \sigma_0, \sigma_0 \cdot \nu) \notin X$, then there is $u_0 \in V$ such that

$$\xi(\nabla u_0, \gamma u_0) = \langle\langle u_0, (-\operatorname{div} \sigma_0, \sigma_0 \cdot \nu)\rangle\rangle < \Phi(\nabla u_0, \gamma u_0).$$

This is a contradiction. Thus $(-\operatorname{div} \sigma_0, \sigma_0 \cdot \nu) \in X$.

(2) Next assume **(H1)**,**(H2′)**,**(C′)** and **(H4)**. We note that there is $(F_0, f_0) \in X$ satisfying $\rho(\nabla u, 0) \geq L_{(F_0,f_0)}(u)$ for all $u \in W^{1,1}(\Omega)$ by the next lemma. Taking constant functions, we see that (F_0, f_0) satisfies the conservation law. By (1) of this proof, there is $\sigma_1 \in L^\infty(\omega; R^n)$ such that $\sigma_1(x) \in \Gamma(x)$ for a.e. $x \in \omega$, $-\mathrm{div}\,\sigma_1 = F_0$ a.e. on ω and $\sigma_1 \cdot \nu = f_0$ H_{n-1}-a.e. on $\partial\omega \cap \partial\Omega$.

We set $\tilde{f}_0 = f_0$ on $\partial\Omega - \partial\omega$ and $\tilde{f}_0 = -\sigma_1 \cdot \nu^\omega$ on $\Omega \cap \partial\omega$, where ν^ω is the unit outer normal to ω. Furthermore let $\tilde{F}_0$ be the restriction of F_0 to $\Omega - \overline{\omega}$. Then $(\tilde{F}_0, \tilde{f}_0)$ satisfies the conservation law on $\Omega - \overline{\omega}$.

It follow that there is $\sigma_2 \in L^\infty(\Omega - \overline{\omega}, R^n)$ such that $-\mathrm{div}\,\sigma_2 = \tilde{F}_0$ a.e. on $\Omega - \overline{\omega}$, $\sigma_2 \cdot \nu = \tilde{f}_0 = f_0$ H_{n-1} a.e. on $\partial\Omega - \partial\omega$ and $\sigma_2 \cdot \nu = \tilde{f}_0 = -\sigma_1 \cdot \nu^\omega$ H_{n-1} a.e. on $\Omega \cap \partial\omega$, since $\Omega - \overline{\omega}$ is admissible.

Now let $\sigma_3 = \sigma_1$ on ω and $\sigma_3 = \sigma_2$ on $\Omega - \omega$. In view of the equality $\sigma_2 \cdot \nu = -\sigma_1 \cdot \nu^\omega$ on $\Omega \cap \partial\omega$ and Green's formula, we can show that $-\mathrm{div}\,\sigma = F_0$ on Ω. Evidently $\sigma_0 \cdot \nu = f_0$ on $\partial\Omega$ and the proof is completed. □

The following lemma is proved in [1]. For the completeness, we give the proof which is slightly different from that in [1].

LEMMA 2.6. *Assume that X is weakly* compact and*

$$\int_\Omega \beta(\nabla u, \cdot)dx \geq \inf_{(F,f)\in X} L_{(F,f)}(u) \qquad \text{for all } u \in W^{1,1}(\Omega).$$

Then there is $(F_0, f_0) \in X$ such that $\int_\Omega \beta(\nabla u, \cdot)dx \geq L_{(F_0,f_0)}(u)$ for all $u \in W^{1,1}(\Omega)$.

Proof. Assume that the conclusion does not hold. Then for each $(F, f) \in X$ there is $u \in W^{1,1}(\Omega)$ such that $\int_\Omega \beta(\nabla u, \cdot)dx < L_{(F,f)}(u)$. Let $G_{(u,\epsilon)} = \{(F, f) \in X;\ \int_\Omega \beta(\nabla u, \cdot)dx - L_{(F,f)}(u) < -\epsilon\}$ for $u \in W^{1,1}(\Omega)$ and $\epsilon > 0$. Then each $G_{(u,\epsilon)}$ is an open subset of X and $\{G_{(u,\epsilon)}\}$ forms a covering of X. Since X is a weak* compact set, there are $(u_1, \epsilon_1), \ldots, (u_t, \epsilon_t)$ such that

$$\cup_{i=1}^t G_{(u_i,\epsilon_i)} \supset X.$$

Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_t\}$ and K_0 be the convex hull of $u_1, \ldots, u_t$. Then $\cup_{i=1}^t G_{(u_i,\epsilon)} \supset X$ so that

$$\sup_{(F,f)\in X} \inf_{u\in K_0} \left(\int_\Omega \beta(\nabla u, \cdot)dx - L_{(F,f)}(u) \right) < -\epsilon.$$

It follows from a min-max theorem that

$$\inf_{u\in K_0}\sup_{(F,f)\in X}\left(\int_\Omega \beta(\nabla u,\cdot)dx - L_{(F,f)}(u)\right) < -\epsilon.$$

Accordingly, there is $u_0 \in K_0$ with $\sup_{(F,f)\in X}\left(\int_\Omega \beta(\nabla u_0,\cdot)dx - L_{(F,f)}(u_0)\right) < -\epsilon < 0$. This is a contradiction. □

We conclude this section with a special case which implies a variant of the supply–demand theorem. Let

$$\lambda(u) = \int_{\partial\Omega} \gamma\chi_S\lambda dH_{n-1}, \quad \mu(S) = \int_{\partial\Omega}\gamma\chi_S\mu dH_{n-1}, \quad F(S) = \int_\Omega \chi_S F dx$$

for $u \in BV(\Omega)$. If $u = \chi_S$, then we denote $\lambda(u), \mu(u), F(u)$ simply by $\lambda(S), \mu(S), F(S)$.

PROPOSITION 2.7. *Let λ, μ be H_{n-1} -measurable functions on $\partial\Omega$, let $F_0 \in L^n(\Omega)$ and let*

$$X = \{(F_0, f);\ \lambda \le f \le \mu\ H_{n-1}\text{-a.e. on } \partial\Omega\}.$$

We assume that **(H1)** *,* **(H2)** *and one of* **(H3)** *and* **(H4)** *in Theorem 2.5 hold. Then condition* **(C)** *is equivalent with*

(CG) *$C(S) \ge \lambda(S) + F_0(S)$ and $C(S) \ge \mu(\Omega - S) - F(\Omega - S)$ for all $S \in Q$.*

Proof. It is easy to see that **(C)** implies **(CG)**. We prove the converse. Let $u \in BV(\Omega)$ and set

$$N_t = \{x \in \Omega;\ u(x) \ge t\}, \quad M_t = \{x \in \Omega;\ u(x) \le t\}$$

for $t \in R$. By [10, Lemmas 4,6, 5.4], we have

$$\begin{aligned}\inf_{(F,f)\in X} L_{(F,f)}(u) &= \int_\Omega uF_0dx + \int_{\partial\Omega} u^-\mu dH_{n-1} + \int_{\partial\Omega} u^+\lambda dH_{n-1}\\ &= \int_0^\infty\left(\int_\Omega \chi_{N_t}F_0dx + \int_{\partial\Omega}\gamma\chi_{N_t}\lambda dH_{n-1}\right)dt\\ &\quad + \int_{-\infty}^0\left(\int_\Omega -\chi_{M_t}F_0dx + \int_{\partial\Omega}\gamma\chi_{M_t}\mu dH_{n-1}\right)dt\end{aligned}$$

with $u^+ = \max(u,0)$ and $u^- = -\min(u,0)$. Furthermore by an equality of coarea formula type [10, Proposition 2.4], we have

$$\psi(u) = \int_{-\infty}^\infty \psi(\chi_{N_t})dt = \int_0^\infty \psi(\chi_{N_t})dt + \int_{-\infty}^0 \psi(-\chi_{M_t})dt.$$

Now assume that **(CG)** holds. Then

$$\begin{aligned}\psi(\chi_{N_t}) &= C(N_t) \geq \int_\Omega \chi_{N_t} F_0 dx + \int_{\partial\Omega} \gamma\chi_{N_t}\lambda dH_{n-1} \\ \psi(-\chi_{M_t}) &= C(\Omega - M_t) \geq \int_\Omega -\chi_{M_t} F_0 dx + \int_{\partial\Omega} \gamma\chi_{M_t}\mu dH_{n-1}.\end{aligned}$$

Integrating both sides, we obtain

$$\psi(u) \geq \inf_{(F,f)\in X} L_{(F,f)}(u).$$

This completes the proof. □

3 Application to a duality of max-flow problems

We apply the feasibility theorem proved in the previous section to a continuous version of max-flow problems (MF). Such problems are introduced by [6] and [13] and developed in [10]. Let X be a subset of $L^n(\Omega) \times L^\infty(\partial\Omega)$. Then (MF) and the dual problem (MF*) are formulated as follows:

(MF) Maximize $\lambda >$ subject to $(-\operatorname{div}\sigma, \sigma\cdot\nu) \in \lambda X, \lambda > 0$, $\sigma \in L^\infty(\Omega; R^n)$ satisfying $\sigma(x) \in \Gamma(x)$ for a.e. $x \in \Omega$.

(MF*) Minimize $\psi(u)/\inf_{(F,f)\in X} L_{(F,f)}(u)$ subject to $u \in W^{1,1}(\Omega)$ and $\inf_{(F,f)\in X} L_{(F,f)}(u) > 0$.

We denote the maximizing value of (MF) by MF and the minimizing value of (MF*) by MF^*. Then we have

THEOREM 3.1. *Assume that* **(H1)** *and* **(H2′)** *holds. Then under one of conditions* **(H3)** *and* **(H4)** *in Theorem 2.5,* $MF = MF^*$ *holds, where we use the convention that the infimum on the empty set is* ∞. *Furthermore* (MF) *has an optimal solution if* MF *is finite.*

Proof. The inequality $MF \leq MF^*$ directly follows from Green's formula. We prove the converse inequality. Let r be an arbitrary positive number equal to or less than MF^*. Then $r\inf_{(F,f)\in X} L_{(F,f)}(u) \leq \psi(u)$ for all $u \in W^{1,1}(\Omega)$ if $\inf_{(F,f)\in X} L_{(F,f)}(u)$ is positive. This inequality trivially holds if $\inf_{(F,f)\in X} L_{(F,f)}(u)$ is nonpositive so that there is a feasible flow σ_0 such that $(-\operatorname{div}\sigma_0, \sigma_0\cdot\nu) \in rX$ by Theorem 2.5. It follows that $r \leq MF$. This shows that $MF^* \leq MF$. If MF is finite, then applying the same argument to $r = MF$ we can prove the existence of optimal solutions to (MF). □

If A and B are disjoint measurable subset of $\partial\Omega$ and

$$X = \{(F, f);\ F = 0 \text{ a.e. on } \Omega, f = 0\ H_{n-1}\text{-a.e. on } \partial\Omega - (A \cup B), \int_A f dH_{n-1} = 1\},$$

then we call (MF) a max-flow problem of Iri's type and denote it by (MFI), or more precisely, ($\mathrm{MFI}_{(A,B)}$).

On the other hand, if $F_0 \in L^n(\Omega)$, $f_0 \in L^\infty(\partial\Omega)$ with the conservation law and $X = \{(F_0, f_0)\}$, then we call (MF) a max-flow problem of Strang's type and denote it by (MFS) or ($\mathrm{MFS}_{(F_0,f_0)}$).

We denote MF^* corresponding to MFI, MFS by $MFI^*_{(A,B)}, MFS^*_{(F_0,f_0)}$ respectively. For such cases, (MF^*) is written in terms of characteristic functions, which we call a continuous version of min-cut problems. Using equalities of coarea formula type as stated in the proof of Proposition 2.7, we can prove the following proposition. (cf. [10].)

PROPOSITION 3.2. *Assume* **(H2)**. *Then*

$$\begin{aligned} MFI^*_{(A,B)} &= \inf\{C(S);\ S \in Q, H_{n-1}(A - \partial^* S) = H_{n-1}(B \cap \partial * S) = 0\}, \\ MFS^*_{(F_0,f_0)} &= \inf\{C(S)/L_{(F_0,f_0)}(\chi_S);\ S \in Q, L_{(F_0,f_0)}(\chi_S) > 0\}. \end{aligned}$$

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